TOPIC 17: TWO PERSON GAMES (MIXED STRATEGY)

We saw in the last section that not every simultaneous move, two person, zee-sum game has a saddle point and that one should consider a mixed strategy in these cases. Because of time constraints, we will limit our discussion to games where both players have two possible strategies. The general principles are the same in games where the players have more strategies and details can be found in Rolf [2] or Gilbert and Hatcher [1].

1. Mixed Strategies.

Definition 1.1. *A* **mixed strategy** *for a player with two strategies, A and B, is a choice of probabilities p_1 and p_2 with $0 \leq p_1, p_2 \leq 1$ and $p_1 + p_2 = 1$. The player selects strategy A with probability p_1 and strategy B with probability p_2. The player should make the choice in random way so that his/her opponent cannot detect a pattern in his/her play (for example, the player might use a device such as the the spinner shown below to select his/her strategy on the next play). The row player's mixed strategy is represented as a row $\begin{pmatrix} p_1, p_2 \end{pmatrix}$ and the column player's mixed strategy is represented as a column $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$*

Example 1.1. *Recall our example from fencing where both players had the option of attacking(A) or holding back (H) at the beginning of each bout with the following pay-off matrix showing the expected number of points for Rhonda for each situation:*

		Cathy	
		A	**H**
Rhonda	**A**	0.5	-0.2
	H	-0.3	0.5

If Rhonda plays the mixed strategy of $\begin{pmatrix} 0.5, 0.5 \end{pmatrix}$, it means that she attacks off the line 50 percent of the time and she holds back 50% of the time. she does this in an unpredictable way, so that her opponent does not know whether she will attack or hold back at the beginning of the next bout. If Cathy plays $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, this means that Cathy attacks off the line 70% of the time and she hold back 30% of the time and her choice for the next bout cannot be predicted by her opponent.

Note that a fixed strategy can be represented as a special case of a mixed strategy. If Rhonda plays $(1, 0)$ above, it means that she always plays strategy A.

2. Deciding between strategies

When deciding between strategies, we compare expected payoff's to make comparisons. For the row player, R, the strategy which maximizes his/her expected payoff should be chosen and for the column player, C, the strategy which minimizes R's expected payoff (maximizes C's expected payoff) is the preferred one.

When both players have only two strategies, calculating the expected payoff for R when you know the strategies for both players is relatively easy. When more strategies are involved it is best to calculate with matrix multiplication.

2.1. Expected Payoff for mixed strategies.

Lets suppose that we have two players, R and C, playing a zero-sum, simultaneous move game. It is implicitly assumed that the players make their choice of strategy independently of each other. Lets assume that R has two strategies $R1$ and $R2$ and C also has 2 strategies $C1$ and $C2$ and that the payoff matrix for R is given by

		C	
		C1	**C2**
R	**R1**	a	b
	R2	c	d

We assume that R is playing the mixed strategy (p_1, p_2) and C is playing the mixed strategy $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Because the players choose their strategies independently the probability that R will choose $R1$ and C will choose $C2$ is $p_1 q_2$ (from our formula for independent events $P(A \cap B) = P(A)P(B)$). Notice that the payoff for R is a random variable, X, and its value depends on which of the four situations occurs. We can find its probability distribution using the fact that the decisions on strategy are made independently:

Choice	X = Pay-off for R	Probability	$XP(X)$
$R1C1$	a	$(p_1)(q_1)$	$a(p_1)(q_1)$
$R1C2$	b	$(p_1)(q_2)$	$b(p_1)(q_2)$
$R2C1$	c	$(p_2)(q_1)$	$c(p_2)(q_1)$
$R2C2$	d	$(p_2)(q_2)$	$d(p_2)(q_2)$
			$E(X) = (ap_1 + cp_2)(q_1) + (bp_1 + dp_2)(q_2)$

Example 2.1. *Consider the example of a zero-sum game from fencing above with payoff matrix*

		Cathy	
		A	**H**
Rhonda	**A**	0.5	-0.2
	H	-0.3	0.5

(a) Assume that Rhonda plays $(0.5, 0.5)$ and Cathy plays $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, calculate the expected payoff for R. What is the expected payoff for C?

(b) Suppose that Cathy continues to play the mixed strategy $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$ and Rhonda switches to the mixed strategy $(.2, .8)$. What is the expected payoff for R?

(c) If Cathy continues to play $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$ which of the above strategies is better for Rhonda?

2.2. **C's best counterstrategy.** Suppose R plays the a mixed strategy $(p, 1-p)$ for some value of p with $0 \leq p \leq 1$.

If C plays the pure strategy $C1$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then R's expected payoff is $J = ap + c(1-p)$.

If C plays the pure strategy $C2$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then R's expected payoff is $K = bp + d(1-p)$.

If C plays the strategy $\begin{pmatrix} q \\ 1-q \end{pmatrix}$, the expected payoff for R is $qJ + (1-q)K$.

Suppose $J < K$, then $qJ+(1-q)K < qK+(1-q)K = K$ and $qJ+(1-q)K > qJ+(1-q)J = J$, giving us that the expected payoff is between J and K, which shows that the best counterstrategy for C is the pure strategy $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Suppose $J > K$, then $qJ+(1-q)K > qK+(1-q)K = K$ and $qJ+(1-q)K < qJ+(1-q)J = J$, giving us that the expected payoff is between J and K, which shows that the best counterstrategy for C is the pure strategy $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The bottom line here is that if R settles on any given mixed strategy $(p, 1-p)$ in the long run, the best counterstrategy for C is a pure strategy. A symmetric argument shows that the same conclusion is true for a best counterstrategy for R.

Example 2.2. *Consider the example of a zero-sum game from fencing above with payoff matrix*

		Cathy	
		A	**H**
Rhonda	**A**	0.5	-0.2
	H	-0.3	0.5

(a) Assume that Rhonda plays $(0.5, 0.5)$, what is the optimal counterstrategy for C?

3. Best Mixed Strategy

Now by our assumptions for both players, the situation is much more dynamic than that described in the previous section. We assume that both players observe the statistics and have complete information about the other player's mixed strategy at all times. We also assume that both adjust their play to the best counterstrategy continuously. One might then question whether equilibrium is possible under these assumptions.

The following theorem is one of the fundamental theorems of game theory and ensures an equilibrium under the above assumptions:

Minimax Theorem: John Von Neumann For every zero sum game, there is a number ν for value and particular mixed strategies for both players such that

(1) The expected payoff to the row player will be at least ν if the row player plays his or her particular mixed strategy, no matter what mixed strategy the column player plays.
(2) The expected payoff to the row player will be at most ν if the column player plays his or her particular mixed strategy, no matter what strategy the row player chooses.

the number ν is called the **value of the game** and represents the expected advantage to the row player (a disadvantage if ν is negative).

If both players play the strategies from the theorem, the system will be in equilibrium, since neither player should be able to increase their payoff by unilaterally changing their strategy. Thus the long run expected payoff for R will be μ and this is the **value** of the game.

3.1. **Strategy Lines and R's optimal mixed strategy.** Given a 2×2 payoff matrix, we can draw a picture of the possible payoffs for R as shown in the example below. We draw lines representing R's payoff for each of C's pure strategies. (This payoff will vary as p varies in R's strategy $[p, 1-p]$.) These lines are called strategy lines.

Example Lets look at an example, where the payoff matrix is given by

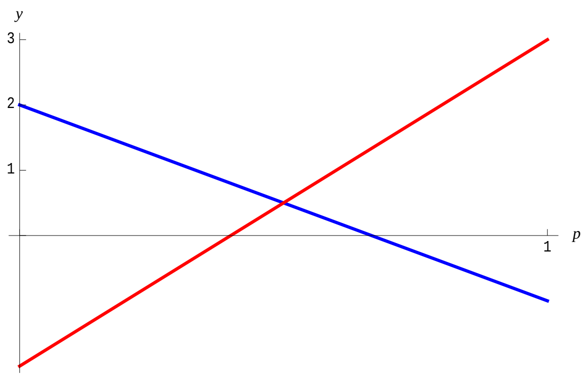
$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}.$$

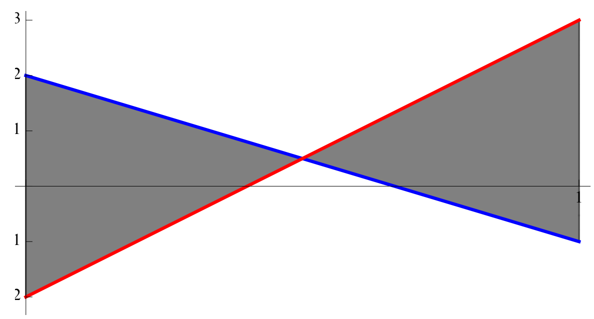
Let $[p, 1-p]$ denote R's strategy. We draw a co-ordinate system with the variable p on the horizontal axis and $y =$ the expected payoff for R on the vertical axis. **The lines shown give the expected payoff for R for the two pure strategies that C might pursue. These are called strategy lines.**

If R plays $[p, 1-p]$ and C plays $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the expected payoff for R is

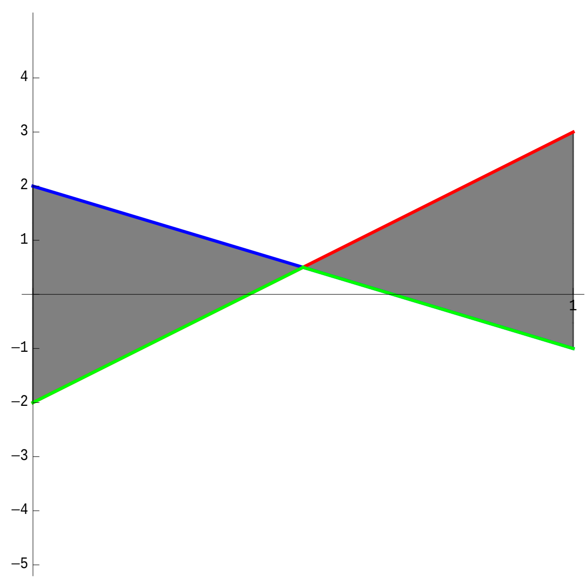
$$J = ap + c(1-p) = -p + 2(1-p) = 2 - 3p$$

is Thus the line showing the expected value for R when C always plays Col. 1 is $\boxed{y = 2 - 3p}$ (shown in blue below).
Similarly, we see that the line showing the expected value for R when C always plays Col. 2 is $\boxed{y = 5p - 2}$ (shown in red below).





Now R can choose the value of p and we assume that C will respond appropriately and choose their best possible counterstrategy. Recall that whatever strategy C chooses, R's expected value will be in the shaded region on the right. The best counterstrategy for C is to choose the pure strategy that will give the minimum expected payoff for R. Hence, **if C chooses the best counterstrategy for any given choice of p made by R, R's payoff will be minimized and will appear along the line highlighted in green below.**



Now since R wants to maximize his/her payoff,R **chooses the strategy corresponding to the value of p which gives the maximum along the green line. This is where the lines in this picture meet.**

Find the value of p for which the above strategy lines meet and find R's best mixed strategy.

General Case If R's payoff matrix has a saddle point, then the lines might not meet or may meet when $p = 0$ or when $p = 1$. Otherwise (**in the case where the strategy matrix is reduced**), we can solve p at the point where both lines meet.

If the strategy matrix is given by

		C	
		C1	**C2**
	R1	a	b
R			
	R2	c	d

and is reduced, the optimal mixed strategy for R is given by $(p, 1-p)$ where p is given by the solution to the equation $ap + c(1-p) = bp + d(1-p)$ that is when

$$p = \frac{d-c}{(a+d)-(b+c)}.$$

By similar reasoning, we get that the optimal mixed strategy for C is given by $\begin{pmatrix} q \\ 1-q \end{pmatrix}$ where

$$q = \frac{d-b}{(a+d)-(b+c)}.$$

Using these optimal mixed strategies for both players, we get that the value of the game is given by

$$\nu = \frac{ad-bc}{(a+d)-(b+c)}.$$

Example 3.1. *In the example from Winston [3] in the last section, the football team on offense had the options of running or passing the ball and the team on defense could choose a run defense or a pass defense. We got the following pay-off matrix for the team on offense using expected gain in yards for each situation:*

		Defense	
		Run Defense	**Pass Defense**
	Run	-5	5
Offense			
	Pass	10	0

(a) Does this matrix have a saddle point?

(b) Find the optimal strategy for both players.

(c) What is the value of the game and what is its interpretation.

Example 3.2. *Recall our example from fencing where both players had the option of attacking(A) or holding back (H) at the beginning of each bout with the following pay-off matrix showing the expected number of points for Rhonda for each situation:*

		Cathy	
		A	**H**
Rhonda	**A**	0.5	-0.2
	H	-0.3	0.5

(a) Find the optimal mixed strategy for both players.

(b) Find the value of the game

3.2. **Constant sum game.** For a constant sum game the calculations are the same for the optimal mixed strategy for both players and the value of the game, ν (which is the expected payoff for R). If the payoffs for R and C add to K, then the long run expected payoff for C is $K - \nu$.

Example 3.3. *http://mindyourdecisions.com/blog/2012/06/19/game-theory-applied-to-basketball-by-shawn-ruminski/.VQUucEjRCg4 Our example of possible endgame strategies for basketball, had a payoff matrix where the payoff for each team was the probability of a win for each team under the given circumstances. This is a* **Constant sum game** *since the probabilities add to 1.*

		Defending	Team
		Defend 2	**Defend 3**
Offense	**Shoot 2**	0.178	0.312
	Shoot 3	0.5	0.228

(a) What is the optimal strategy for both teams here?

(b) What is the value of the game?

(c) What is the expected payoff for the Offense?

(d) What is the expected payoff for the Defence?

(d) What do these numbers mean?

Example 3.4. *A baseball pitcher throws three pitches, a fastball, a slider and a change-up. As a measure of the payoff for this type of confrontation, we might use the expected number of runs the batter creates in each situation. (Note their are other possible measures that take into account the pitcher's abilities). We would expect that for any given pitch, the batter's performance is better if he anticipates the pitch. Lets assume that the batter has four possible strategies, To anticipate either a fastball, a slider or a change-up or not to anticipate any pitch.*

		Pitcher		
		Fastball	**Change-up**	**Slider**
Batter	**Fastball**	0.38	0.37	0.39
	Change-up	0.25	0.4	0.41
	Slider	0.35	0.32	0.45
	None	0.38	0.3	0.42

(a) Find the reduced payoff matrix.

(b) Find the optimal mixed strategy for both players.

(c) Find the value of the game.

(d) Interpret the value of the game.

Example 3.5. *In a simplified model of a tennis serve, the server must decide whether to serve to the receiver's forehand or to the backhand. The receiver must anticipate whether the serve will come to the forehand or the backhand. The payoff matrix shown below, shows the percentage of time the server (row player) wins the serve in each situation. Note this is a constant sum game with the payoff for the receiver equal to 100 minus the payoff for the server.*

		Receiver	
		Guess Forehand	**Guess Backhand**
Server	**Forehand**	40	70
	Backhand	80	60

(a) Find each player's optional strategy given that both players play optimally.

(b) Find the value of the game.

References

1. Hatcher R Gilbert, G, *Mathematics beyond the numbers*, Wiley (2000).
2. Howard Rolf, *Finite mathematics*, Cengage Learning.
3. Wayne Winston, *Mathletics*, Princeton University Press.